Def $f \in \mathcal{M}(\mathbb{R})$

A function $f: \mathbb{R} \to \mathbb{C}$ is said to be of moderate decrease if (1) f is cts

(2) $\exists\ A > 0$ s.t. $|f(x)| \le \frac{A}{1+x^2}$.

Def

Let $f \in \mathcal{M}(\mathbb{R})$. The Fourier transformation of f is $\hat{f}(\xi) := \int_{-\infty}^{\infty} f(x) e^{-2\pi i \xi x} dx$.

Def

$f, g \in M(\mathbb{R})$. $f * g(x) := \int_{-\infty}^{\infty} f(x-y) g(y) dy$.

Thm (Fourier Inversion Formula)

If $f, \hat{f} \in M(\mathbb{R})$, then $f(x) = \int_{-\infty}^{\infty} \hat{f}(\xi) e^{2\pi i \xi x} d\xi$.

- Give a simplified proof of Fourier Inversion Formula if we further assume f is compactly supported, i.e. $f \equiv 0$ on $\mathbb{R} \setminus [-M, M]$.

- Find f s.t.

$$\int_{-\infty}^{\infty} f(x-y) e^{-|y|} dy = 2e^{-|x|} - e^{-2|x|}.$$

- If $f, \hat{f} \in M(\mathbb{R})$ and $f \equiv 0$ on $\mathbb{R} \setminus [-M, M]$, then $f(x) = \int_{-\infty}^{\infty} \hat{f}(\xi) e^{2\pi i \xi x} dx$.

Step 1: $\forall \delta < \frac{1}{2M}$, $\forall x \in [-\frac{1}{2\delta}, \frac{1}{2\delta}]$,

$$f(x) = \delta \sum_{n=-\infty}^{\infty} \hat{f}(n\delta) e^{2\pi i n \delta x}.$$

Pf: Let $h(x) = f(x)$ on $x \in [-\frac{1}{2\delta}, \frac{1}{2\delta}]$ and extend h to a $\frac{1}{\delta}$-periodic function.

$$\hat{h}(n) = \delta \int_{-\frac{1}{2\delta}}^{\frac{1}{2\delta}} f(x) e^{-2\pi i \delta n x} dx$$

$$= \delta \int_{-\infty}^{\infty} f(x) e^{-2\pi i \delta n x} dx \quad \left(\begin{array}{l} \frac{1}{2\delta} > M \\ f(x) = 0 \text{ on } \mathbb{R} \setminus [-M, M] \end{array} \right)$$

$$= \delta \hat{f}(n\delta)$$

If we can show $h(x) = \sum_{n=-\infty}^{\infty} \hat{h}(n) e^{2\pi i \delta n x}$,

then $f(x) = \delta \sum_{n=-\infty}^{\infty} \hat{f}(n\delta) e^{2\pi i \delta n x}$, $\forall x \in [-\frac{1}{2\delta}, \frac{1}{2\delta}]$.

It suffices to show

$$\sum_{n=-\infty}^{\infty} |\hat{h}(n)| < \infty, \text{ i.e. } \delta \sum_{n=-\infty}^{\infty} |\hat{f}(n\delta)| < \infty.$$

Since $\hat{f} \in \mathcal{M}(\mathbb{R})$, $|\hat{f}(n\delta)| \leq \frac{A}{1+(n\delta)^2}$.

$$\delta \sum_{n=-\infty}^{\infty} |\hat{f}(n\delta)| \leq \delta \sum_{n=-\infty}^{\infty} \frac{A}{1+\delta^2 n^2} < \infty$$

#

Step 2: $\forall F \in \mathcal{M}(\mathbb{R})$,

$$\int_{-\infty}^{\infty} F(\xi) d\xi = \lim_{\delta \to 0^+} \delta \sum_{n=-\infty}^{\infty} F(\delta n).$$

Pf: $\int_{-\infty}^{\infty} F(\xi) d\xi = \lim_{N \to \infty} \int_{-N}^{N} F(\xi) d\xi$.

$\forall N$, Take N_0 s.t. $N_0 \delta \leq N < (N_0+1)\delta$.

We consider the partition

$$\mathcal{P} = \{-N, -N_0\delta, -(N_0-1)\delta, \cdots, 0, \delta, \cdots, N_0\delta, N\}$$

Then $\displaystyle\int_{-N}^{N} F(\xi)\,d\xi = \lim_{\delta\to 0^+} \Big(\delta \sum_{n=-N_0}^{N_0} F(\delta n) + (N-N_0\delta)F(N) \Big)$

Since $N - N_0\delta < \delta$ and $|F(N)| < C$,

then $\displaystyle\lim_{\delta\to 0^+} (N-N_0\delta)F(N) \le \lim_{\delta\to 0^+} C\delta = 0.$

Thus $\displaystyle\int_{-N}^{N} F(\xi)\,d\xi = \lim_{\delta\to 0^+} \delta \sum_{n=-N_0}^{N_0} F(\delta n).$

$$= \lim_{\delta\to 0^+} \delta \sum_{|n|\le\frac{N}{\delta}} F(\delta n).$$

$$\Big|\delta \sum_{|n|>\frac{N}{\delta}} F(\delta n)\Big| \le \delta \sum_{|n|>\frac{N}{\delta}} |F(\delta n)|$$

$$\le \delta \sum_{|n|>\frac{N}{\delta}} \frac{A}{1+(\delta n)^2}$$

$$< \delta \sum_{|n|>\frac{N}{\delta}} \frac{A}{\delta^2 n^2}$$

$$= \frac{A}{\delta} \sum_{|n|>\frac{N}{\delta}} \frac{1}{n^2}$$

$$\le \frac{\tilde{C}}{\delta} \frac{1}{\frac{N}{\delta}}$$

$$= \frac{\tilde{C}}{N}$$, $\tilde{C}$ independent N, δ

$\forall \varepsilon > 0$,

$$\left| \int_{-\infty}^{\infty} F(\xi)\,d\xi - \delta \sum_{n=-\infty}^{\infty} F(n\delta) \right|$$

$$\leq \left| \int_{-\infty}^{\infty} F(\xi)\,d\xi - \int_{-N}^{N} F(\xi)\,d\xi \right| \qquad (\mathrm{I})$$

$$+ \left| \int_{-N}^{N} F(\xi) - \delta \sum_{|n| \leq \frac{N}{\delta}} F(n\delta) \right| \qquad (\mathrm{II})$$

$$+ \left| \sum_{|n| > \frac{N}{\delta}} F(n\delta) \right| \qquad (\mathrm{III})$$

We first take N_1 large s.t. $(\mathrm{I}), (\mathrm{III}) < \varepsilon$.

Then we choose $\delta_0 := \delta_0(N_1)$ s.t. $\forall\ 0 < \delta < \delta_0$,

$(\mathrm{II}) < \varepsilon$.

#

Step 3: $\int_{-\infty}^{\infty} \hat{f}(\xi) e^{2\pi i x \xi}\,dx$

$$= \lim_{\delta \to 0^+} \delta \sum_{n=-\infty}^{\infty} \hat{f}(n\delta)\, e^{2\pi i n\delta x} \qquad \left(\begin{array}{l} F(\xi) = \hat{f}(\xi) e^{2\pi i \xi x} \\ \text{Apply step 2} \end{array} \right)$$

$$= \hat{f}(x), \quad \forall\ x \in \left[-\tfrac{1}{2\delta}, \tfrac{1}{2\delta}\right].$$

Since δ can be arbitrarily small,

$$\hat{f}(x) = \int_{-\infty}^{\infty} \hat{f}(\xi) e^{2\pi i x \xi}\,dx, \quad \forall\ x \in \mathbb{R}.$$

□

· Find f s.t.

$$\int_{-\infty}^{\infty} f(x-y)e^{-|y|}dy = 2e^{-|x|} - e^{-2|x|}.$$

Pf: Define $g(x) = e^{-|x|}$. Then

$$L.H.S. = f*g(x).$$

To find f, we first find $\hat{f}$ and then apply Fourier Inversion Formula.

Taking Fourier transformation,

$$L.H.S. = \widehat{f*g}(\xi) = \hat{f}(\xi)\hat{g}(\xi)$$

If we know $\widehat{f*g}$ and $\hat{g}$, then we get $\hat{f}$.

$$\hat{g}(\xi) = \int_{-\infty}^{\infty} e^{-|x|}e^{-2\pi i\xi x}dx$$

$$= \int_0^{\infty} e^{-x}e^{-2\pi i\xi x}dx + \int_{-\infty}^{0} e^{x}e^{-2\pi i\xi x}dx$$

$$= \frac{1}{1+2\pi i\xi} + \frac{1}{1-2\pi i\xi}$$

$$= \frac{2}{1+4\pi^2\xi^2}.$$

$$f*g(x) = 2e^{-|x|} - e^{-2|x|}.$$

Let $h(x) = e^{-2|x|} = g(2x)$.

$$\hat{h}(\xi) = \frac{\hat{g}(\frac{\xi}{2})}{2} = \frac{1}{2}\,\frac{2}{1+4\pi^2(\frac{\xi}{2})^2}$$

$$= \frac{1}{1+\pi^2\xi^2}.$$

$$\widehat{f*g}(\xi) = \frac{4}{1+4\pi^2\xi^2} - \frac{1}{1+\pi^2\xi^2}$$

$$\hat{f}(\xi) = \frac{1+4\pi^2\xi^2}{2}\left(\frac{4}{1+4\pi^2\xi^2} - \frac{1}{1+\pi^2\xi^2}\right)$$

$$= 2 - \frac{1+4\pi^2\xi^2}{2(1+\pi^2\xi^2)}$$

$$= \frac{4+\cancel{4\pi^2\xi^2}-1-\cancel{4\pi^2\xi^2}}{2(1+\pi^2\xi^2)}$$

$$= \frac{3}{2}\,\frac{1}{1+\pi^2\xi^2}$$

$$= \frac{3}{2}\hat{h}(\xi).$$

By Fourier Inversion Formula,

$$f(x) = \frac{3}{2}h(x) = \frac{3}{2}e^{-2|x|}. \qquad \square$$

Rmk: If $f, \hat{f} \in \mathcal{M}(\mathbb{R})$, f is even,

$\hat{\hat{f}}(x) = f(x)$.

$$f(x) = \int_{-\infty}^{\infty} \hat{f}(\xi) e^{2\pi i \xi x} d\xi$$

$$= \int_{\infty}^{-\infty} \hat{f}(-\eta) e^{-2\pi i \eta x} (-d\eta) \quad (\eta = -\xi)$$

$$= \int_{-\infty}^{\infty} \hat{f}(-\eta) e^{-2\pi i \eta x} d\eta$$

$$= \int_{-\infty}^{\infty} \hat{f}(\eta) e^{-2\pi i \eta x} d\eta$$

$$= \hat{\hat{f}}(x).$$

$\square$